PMM U.S.S.R.,Vol.49,No.4,pp.527-529,1985
Printed in Great Britain

0021-8928/85 $10.00+0.00
Pergamon Journals Ltd.

NATURAL CONVECTION IN LIQUIDS CAUSED BY ABSORPTION OF LASER RADIATION*

R.S. AKOPYAN and B.YA. ZEL'DOVICH

A problem of regular, natural convection in a horizontal layer of liquid in a gravitational field, with volume heat sources distributed in the plane of the layer in a spatially periodic manner, is considered. It is shown that the response of the system is greatest, other conditions being equal, when the period of the source is approximately equal to twice the thickness of the layer.

The literature offers a large number of analytically solved problems on natural convection when there is a temperature gradient /1-3/ created when heat is applied to the layer boundaries. At present, the absorption of energy from a coherent light wave generated by a laser makes it possible to produce volume heat emission with practically any spatial distribution required, and to alter this distribution at will without any difficulties.

1. Linearized convection equations with volume heat sources. Let us consider a horizontal layer of liquid $-L/2 \leqslant z \leqslant L/2$ of thickness L, in the gravity field $\mathbf{g} = -g\mathbf{e}_z$, $g > 0$. We shall assume that two, plane coherent light waves impinge on the layer, and their interference leads to a spatially periodic intensity distribution $|E(x, y)|^2$ and, with weak absorption of light, to volume-distributed heat sources of the form

$$Q(x, y) = \frac{\varkappa cn}{8\pi}|E(x, y)|^2 = \frac{\varkappa cn}{8\pi}[|E_1|^2 + |E_2|^2 + E_1 E_2^* \exp(ik_x x + ik_y y) + E_1^* E_2 \exp(-ik_x x - ik_y y)] \tag{1.1}$$

Here $k = (k_x, k_y)$ is the wave vector of the inhomogeneous part of the heat emission, $|k| = 2\pi|\sin\alpha_1 - \sin\alpha_2|/\lambda$ where α_1, α_2 are the angles of incidence of the waves, λ is the wavelength of the light in air, $\varkappa$ is the light absorption coefficient $(\varkappa L \ll 1)$, c is speed of light in vacuo, and n is the refractive index of the liquid.

Let the temperature T_0 be maintained at the rigid boundaries $z = \pm L/2$ of the layer, and the boundary condition of adhesion of the liquid $\mathbf{v}(z = \pm L/2) = 0$ be given. We have accordingly the equilibrium state of the light field

$$\mathbf{v}_0 = 0,\ T_0 = \text{const},\ \rho_0 = \text{const},\ p_0 = p(z = 0) - \rho_0 g z$$

where $\mathbf{v}$ is the velocity, ρ is the density and p is the pressure. When the layer is iluminated, the system becomes perturbed and the steady state equations for the variations $\theta = T - T_0$, $\delta\rho = -\alpha\rho_0\theta$, $\delta p = p - p_0$ have, in the Boussinesq approximation, the form /1, 2/

$$\nabla^2\theta = -q|E(x, y)|^2,\quad q = \frac{\varkappa cn}{8\pi\rho_0 c_p \chi} \tag{1.2}$$

$$\eta\nabla^2\mathbf{v} - \operatorname{grad}\delta p - \rho_0 g\alpha\theta\mathbf{e}_z = 0 \tag{1.3}$$

$$\operatorname{div}\mathbf{v} = 0 \tag{1.4}$$

Here c_p is the heat capacity, χ is the thermal diffusivity, η is viscosity and α is the volume expansion coefficient of the liquid.

Applying as usual the operators $\mathbf{e}_z \operatorname{rot}$ and $\mathbf{e}_z \operatorname{rot}\operatorname{rot}$ to Eq.(1.3) and using (1.4), we obtain

$$\nabla^2\xi = 0,\ \xi \equiv \mathbf{e}_z \operatorname{rot}\mathbf{v} \tag{1.5}$$

$$\nabla^4 v_z + A\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) = 0,\quad A = \frac{\alpha\rho_0 g}{\eta} \tag{1.6}$$

From the boundary conditions it follows that $\partial v_x/\partial x = \partial v_x/\partial y = \partial v_y/\partial y = \partial v_y/\partial x = 0$ when $z = \pm L/2$. This, together with (1.4), implies that $\partial v_z/\partial z = 0$, $\xi(\mathbf{r}) = 0$ when $z = \pm L/2$. The system of equations (1.1), (1.2), (1.4)-(1.6) and the boundary conditions shown are used to determine the perturbed quantities $\theta(\mathbf{r})$ and $\mathbf{v}(\mathbf{r})$.

2. Natural convection. From (1.5) and the boundary conditions for ξ it follows that $\xi(\mathbf{r}) = 0$ at all points of the liquid, i.e. there are no "screw" motions. We shall seek the solution of the system of equations (1.2), (1.4), (1.6) and $\xi(\mathbf{r}) = 0$ in the form

$$\theta(\mathbf{r}) = q\frac{|E_1|^2 + |E_2|^2}{2}\left[\left(\frac{L}{2}\right)^2 - z^2\right] + \Theta(z)\exp(ik_x x + ik_y y) + \text{c.c.} \tag{2.1}$$

$$v_{x,y,z}(\mathbf{r}) = V_{x,y,z}(z)\exp(ik_x x + ik_y y) + \text{c.c.}$$

*Prikl.Matem.Mekhan.,49,4,685-686,1985

Then we obtain the following system of equations for $\Theta(z)$ and $V_z(z)$:

$$\left(\frac{d^2}{dZ^2}-a^2\right)\Theta=-\frac{1}{L^2A}I,\quad \left(\frac{d^2}{dZ^2}-a^2\right)^2 V_z=L^2Aa^2\Theta \tag{2.2}$$

$$a=kL,\quad k^2=k_x^2+k_y^2,\quad Z=\frac{z}{L},\quad I=\frac{\alpha g L^4 \varkappa c n E_1 \bar{E}_2}{8\pi c_p \varkappa\eta}$$

The spatially homogeneous part of the heat emission leads, neglecting convection, to the parabolic temperature distribution described by formula (2.1). The temperature gradient for this distribution is greatest near the boundary, where $|\partial T/\partial z|=q(|E_1|^2+|E_2|^2)/2$. Introducing now the Rayleigh number with help of the above value of the temperature gradient

$$R=\frac{L^4}{\varkappa}\left|\frac{\partial T}{\partial z}\right|=\frac{\alpha g L^5 \varkappa c n}{16\pi c_p \varkappa^2\eta}\left(|E_1|^2+|E_2|^2\right) \tag{2.3}$$

we can formulate the criterion of applicability of the above arguments. The linearization of the initial system leading to the Boussinesq approximation is valid for $R\leqslant 4\cdot 10^3$. As we know (e.g. /4/), the value $R=R_+=18{,}66\cdot 10^3$ determines the threshold of stability in the case when the upper and lower boundary of the layer are at the same temperature and the unstable stratification depends only on internal evolution of heat. Therefore the influence of the homogeneous heat emission on the natural convection discussed here is vanishingly small, provided that the Rayleigh number satisfies, with a margin, the condition given above for the Boussinesq approximation to be applicable.

From (2.2) we obtain the following equation and boundary conditions for the z-component of the velocity:

$$\left(\frac{d^2}{dZ^2}-a^2\right)^3 V_z=-Ia^2 \tag{2.4}$$

$$Z=\pm\frac{1}{2},\quad V_z=\frac{dV_z}{dZ}=\left(\frac{d^2}{dZ^2}-a^2\right)^2 V_z=0 \tag{2.5}$$

As we see, Eq.(2.4) and boundary conditions (2.5) are invariant under the transformation $Z\to -Z$, therefore its solution must be an even function of Z. Therefore the general solution of (2.4) has the form

$$V_z(Z)=Ia^{-4}+(c_1+c_3Z^2)\,\mathrm{ch}\,(aZ)+c_2Z\,\mathrm{sh}\,(aZ) \tag{2.6}$$

The boundary conditions at $Z={}^1/_2$ and $Z=-{}^1/_2$ are identical, yielding three conditions (2.5) for determining three constants c_1, c_2, c_3, and we obtain

$$c_1=-\frac{I}{32a^4}\left[32a\,\mathrm{ch}^2\frac{a}{2}+(32+a^2)\,\mathrm{sh}\,a-a^3\right]\left[\mathrm{ch}\,\frac{a}{2}\,(a+\mathrm{sh}\,a)\right]^{-1} \tag{2.7}$$

$$c_2=\frac{I}{4a^3}\left(8\,\mathrm{sh}\,\frac{a}{2}+a\,\mathrm{ch}\,\frac{a}{2}\right)(a+\mathrm{sh}\,a)^{-1},\quad c_3=-\frac{I}{8a^2}\left(\mathrm{ch}\,\frac{a}{2}\right)^{-1}$$

Using the same notation we obtain from (1.3), the first equation of (2.2) and $\xi(\mathbf{r})=0$,

$$\Theta(Z)=\frac{\eta}{\alpha g \rho_0 L^2}\left[Ia^{-2}-8c_3\,\mathrm{ch}\,(aZ)\right] \tag{2.8}$$

$$V_{x,y}(Z)=i\,\frac{a_{x,y}}{a^2}\,\frac{dV_z(Z)}{dZ} \tag{2.9}$$

Rotating the coordinate axes in the x,y plane we can obtain $a_y=0$, and this will be assumed to hold below. Let us consider some special features of the behaviour of the functions obtained. We see from (2.6) that the amplitude of the z-component of the velocity $V_z(Z)$ attains its maximum value V_z^+ at the centre of the cell $V_z^+\equiv V_z(0)$. The latter depends strongly on L, k and $\varkappa L$. For small values of the parameter $a\ll\pi$, i.e. when the interference pattern is the smoothest, when $\Lambda\equiv 2\pi/k\gg L$, $V_z(0)$ behaves like $\propto k^2L^5(\varkappa L)$ $(V_z(0)\approx(13\cdot 2^{-6}/6!)\,Ia^2$ and like $\propto k^{-4}L^{-1}(\varkappa L)$ $(V_z(0)\approx Ia^{-4})$ when $a\gg\pi$. For fixed L and $\varkappa L$, this function of k reaches its maximum value $V_z^+(0)\approx 8{,}19\cdot 10^{-4}I$ when $kL\approx\pi$, which corresponds to the spatially periodic structure with period equal to twice the thickness.

Fig.1 (curve *1*) shows this relationship normalized to unity at the maximum. For fixed k and $\varkappa L$ the function $V_z(0)$ of L assumes its maximum value $V_z^+(0)\approx 0.094\cdot Ia^{-3}$ at $kL\approx 8.2$. Fig.1 (curve *2*) depicts this relationship.

The function $V_z(Z)$ is even about the middle of the layer $z=0$, and the function $V_x(Z)$ is odd. At $a\ll\pi$ the profile of the function $V_z(Z)$ is independent of the parameters of the medium, and of the spatial period of heat emission, and is represented by curve *1* of Fig. 2. When $a\gg\pi$, the amplitude of the z-component of the velocity $V_z(Z)$ is almost constant along the cell, and falls sharply to zero only towards the edges. Curve *2* of Fig.2 is constructed for $a=50$.

The amplitude of the x-component of the velocity is $-iV_x(Z)$. The factor i in (2.9) means that it differs in phase from the z-component of the velocity by $\pi/2$. Let us denote by $\pm Z_+$ the position of two maxima of the modulus of $V_x(Z)$. Then at $a \ll \pi$ we have $Z_+ \approx 0.281$. Neither Z_+ nor the profile $V_x(Z)$ (curve 3 in Fig.2) depend on the parameters of the problem. The value of the maximum behaves like $|V_x(Z_+)| \approx 8.7\cdot 10^{-4}\cdot Ia \propto kL^4\,(\varkappa L)$. In the other limiting case $a \gg \pi$ and the maximum point of the profile $V_x(Z)$ comes into contact with the layer boundaries according to the law $Z_+ \approx 1/2 - 2/a$. The value of the maximum behaves as $|V_x(Z_+)| \approx I/(2a^4e^2) \propto k^{-4}L^{-1}\,(\varkappa L)$. The profile of the function $-iV_x(Z)$ for $a = 50$ is shown in Fig.2 (curve 4).

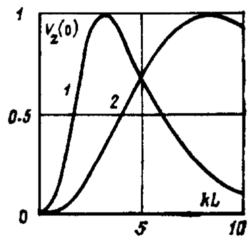

Fig.1

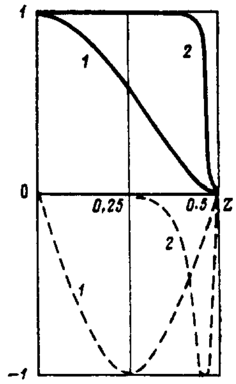

Fig.2

The amplitude of the temperature perturbation $\Theta(Z)$ is an even function of Z and does not depend, in the approximation used here, on whether the liquid moves or not. When $a \ll \pi$, it is parabola with a maximum at $Z = 0$ equal to $\Theta(0) \simeq \eta I/(8\alpha g\rho_0 L^2) \propto L\,(\varkappa L)$, with the profile independent of the parameters of the system. We note that when $a \ll \pi$, the temperature perturbation does not depend on the period of the interference intensity pattern. When $a \gg \pi$, we have $\Theta(0) \approx \eta I/(\alpha g\rho_0 L^2 a^2) \propto k^{-2}L^{-1}\,(\varkappa L)$ and the function $\Theta(Z)$ is almost constant along Z and falls sharply to zero only towards the boundaries.

These results show that already at very moderate values of the power density of the interfering light waves (of the order of 100 w/cm^2 when the thickness is $L \propto 0.1$ cm and $\varkappa L = 0.5$), very strong forced convection takes place. The amplitude of these convective motions ($v_z \propto 0.04$ cm/sec) is clearly sufficient to give the system a required structure of initial perturbations. We should stress that it is very easy, using the light field, to generate initial perturbations of widely differing structure, i.e. in the form of uniform rollers, rollers with dislocations /5/, annular rollers, cells with square or hexagonal packing, perfect or containing various dislocations. A smooth change of the periods of the forced convection patterns is also possible. All this show that laser beams are very suitable for use in the study of convection.

The authors became interested in the problems discussed here in connection with an analogous problem of convection in an anisotropic liquid (in a nematic liquid crystal /4/). In the case of an anisotropic liquid an explicit analytic solution of the convection problem could not be obtained under these conditions. The results obtained above represent one of many examples of an analytic solution of the convection problem in a layer, in which the real boundary conditions are taken into account.

The authors thank V.A. Gorodtsov, V.M. Entov, A.V. Sukhov and Yu.S. Chilingaryan for useful discussions.

REFERENCES

1. CHANDRASEKHAR S., Hydrodynamic and hydromagnetic stability. Oxford. Clarendon Press, 1961.
2. GERSHUNI G.Z. and ZHUKHOVITSKII E.M., Convective Stability of an Incompressible Fluid. Moscow, Nauka, 1972.
3. GERSHUNI G.Z. and ZHUKHOVITSKII E.M., Convective stability. Itogi Nauki i tekhniki. Seriya "Mekhanika zhidkosti i gaza". Moscow, VINITI, 11, 1978.
4. AKOPYAN R.S. and ZEL'DOVICH B.YA., Reorientation of the liquid crystal director with light, near the threshold of a spatially periodic convective instability. Zh. Eksp. Teor. Fiz., 86, 2, 1984.
5. BARANOVA N.B., ZEL'DOVICH B.YA., MAMAEV A.V., PILIPETSKII N.F. and SHKUNOV V.V., Study of the dislocation density of the wave front of light fields with a speckle structure, Zh. Eksp. Teor. Fiz., 83, 5(11), 1982.

Translated by L.K.